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Higher-Order Theory of Gravitation

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The field equations obtained by introducing a correction in the Hilbert Lagrangian in the form of a series of finite terms in R ($\equiv g_{\mu\nu}R^{\mu\nu}$) are considered in order to study the implications for the cosmological singularity.

1. INTRODUCTION

The observable universe today seems to be remarkably homogeneous and isotropic on a very large scale, and a good cosmological model has been constructed which is capable of describing its large-scale properties very nicely. This is the so-called Friedman-Robertson-Walker (FRW) cosmology. It represents a perfectly homogeneous and isotropic space. Regardless of this feature, we have to ask, why is the universe described by such a model? The answers are: either the universe has always been like this—that is, the initial conditions were such that the universe was and has remained isotropic and homogeneous; or the universe started in a less symmetrical phase and evolved through some dynamical process to become an FRW cosmology today. The former is certainly a very unsatisfactory solution, and is also improbable statistically.

A way out is provided by the inflationary scenarios (Guth, 1981). In these models one usually assumes that the universe becomes dominated by a positive vacuum energy—that is, a cosmological constant $\Lambda > 0$—and for a period of time it expands exponentially at the Hubble rate $H = (3\Lambda)^{1/2}$, followed by a reheating period. This eventually terminates in an FRW, flat, radiation-dominated universe. Thereafter, it can proceed with its evolution in the standard way.

At classical and at quantum levels the smoothening of initial inhomogeneities and anisotropies is associated with particle creation

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(Grishchuck and Zel'dovich, 1982). Graviton creation is understood from the fact that the gravitational wave equations, which are an inevitable consequence of Einstein theory of gravitation, are not conformally invariant. This leads to the fact that gravitational waves can be amplified under certain adiabatic conditions in the early universe (Grishchuk, 1977). It amounts to graviton creation in the quantum sense. But this is not so for electromagnetism.

Gravitational wave equations for an FRW background in the form

$$ds^2 = a^2(\eta)(d\eta^2 - dx^2 - dy^2 - dz^2) \tag{1a}$$

$$c\,dt = a(\eta)\,d\eta \tag{1b}$$

take the form

$$\mu'' + \mu[n^2 - (a''/a)] = 0 \tag{2}$$

when transformed conformally ($\bar{g}_{\mu\nu} = e^{-2\sigma} g_{\mu\nu}$; $\bar{\phi}_{\alpha\beta\ldots} = e^{-\sigma(s-1)} \phi_{\alpha\beta\ldots}$) on a flat background (Pandey, 1983). Interestingly, the manifestation of gravitation appears in the form of an additional potential, a''/a. Significantly, it is different from other massless wave equations in a Minkowski background. In order to put this equation on a par with electromagnetism, we considered a modification of the Einstein field equations with the help of a Lagrangian (Pandey, 1983)

$$\mathscr{L} = R - \sum_{n=2}^{N} C_n (l^2 R)^n / 6l^2] \tag{3}$$

where l is the characteristic length and C_n are arbitrary dimensionless coefficients introduced to nullify the addition potential in (2).

This Langragian strongly modifies the Einstein field equations. However, the idea of extending the Einstein-Hilbert action of gravity to include higher derivatives has been around for decades (Weyl, 1919; Pauli, 1919; Eddington, 1965; Yang, 1974). Also, the motivation for examining such extended Einstein theories has been due to their renormalizability and their being ghost-free (Stelle, 1977). They have also been predicted as the low-energy limit of superstring theory (Zwiebach, 1981).

Nonetheless, relativistic gravitational theories derived from a Langrangian containing quadratic terms in the curvature tensor have been considered in constructing cosmological models without singularities (Ruzmaikina and Ruzmaikin, 1970; Nariai and Tomita, 1971; Anderson, 1983, 1984). Therefore, it is natural to study the field equations obtained from the Lagrangian (3) with possible implications for the cosmological singularity.

2. FIELD EQUATIONS

The Lagrangian (3) contains a polynomial in R of a finite number of terms. This should not be disturbing, because it is an observational fact that our universe is not asymptotically flat. There is enough matter on our past light-cone to cause it to refocus (Hawking and Penrose, 1970). The total energy of the universe is exactly zero, the positive energy of the gravitons and matter particles being exactly compensated by the negative gravitational potential energy. That is why the universe is expanding. Also, the unitarity is not well defined, except in scattering calculations in asymptotically flat spaces. Moreover, with these terms it may be possible to find a higher dimensional theory to cancel divergences or to lead to a renormalizable theory. So, we consider the above Langrangian in the form

$$\mathscr{L} = R + \sum_{n=2}^{N} a_n R^n \tag{4}$$

The classical field equations derived from (4) are

$$\begin{aligned} R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \sum_{n=2}^{N} n a_n R^{n-1} [R_{\mu\nu} - (1/2n) g_{\mu\nu} R - [(n-1)/R](R_{;\mu;\nu} - g_{\mu\nu} \Box R) \\ - [(n-1)(n-2)/R^2](R_{;\mu} R_{;\nu} - g_{\mu\nu} R_{;\alpha} R^{;\alpha})] = \kappa T_{\mu\nu} \end{aligned} \tag{5}$$

Now, for the sake of brevity, we drop the summation sign $\sum$, which is indicated by n itself.

The trace is

$$\begin{aligned} &3n(n-1) a_n R^{n-2} \Box R + 3n(n-1)(n-2) a_n R^{n-3} R_{;\alpha} R^{;\alpha} \\ &\quad = \kappa T + R + (2-n) a_n R^n \end{aligned} \tag{6}$$

Equation (5) can be written in the form

$$\begin{aligned} &(1 + n a_n R^{n-1}) R_{\mu\nu} \\ &\quad = x T_{\mu\nu} + n(n-1) a_n R^{n-2} R_{;\mu;\nu} \\ &\qquad + n(n-1)(n-2) a_n R^{n-3} R_{;\mu} R_{;\nu} \\ &\qquad + [(\tfrac{1}{2} R) + \tfrac{1}{2} a_n R^n) - n(n-1) a_n R^{n-2} \Box R - n(n-1) \\ &\qquad \times (n-2) a_n R^{n-3} R_{;\alpha} R^{;\alpha}] g_{\mu\nu} \end{aligned} \tag{7}$$

Here it should be noticed that $1 + n a_n R^{n-1} \neq 0$, or, equivalently,

$$1 + 2a_2 R + 3a_3 R^2 + 4a_4 R^3 + \cdots + N a_N R^{N-1} \neq 0 \tag{8}$$

because of the Cauchy problem.

3. ENERGY CONDITIONS AND THE COSMOLOGICAL MODEL

It is believed that in higher order theories cosmological models can be constructed without singularities. But, as is evident from the Hawking-Penrose theorem, singularities occur under certain conditions. One of these is

$$R_{\mu\nu}\omega^{\mu}\omega^{\nu} \geq 0 \tag{9}$$

for every nonspacelike geodesic; ω^{μ} is the tangent vector. This relation is called the "timelike convergence condition" for timelike geodesics. It comes from the Einstein equations with the "strong energy condition,"

$$T_{\mu\nu}\omega^{\mu}\omega^{\nu} - (T/2)\omega_{\mu}\omega^{\mu} \geq 0 \tag{10}$$

where T is the trace of the energy-momentum tensor $T_{\mu\nu}$.

Again, for null geodesics, the condition is called the "null-convergence condition," which gives, through the Einstein equations, the "weak" energy condition"

$$T_{\mu\nu}\omega^{\mu}\omega^{\nu} \geq 0 \tag{11}$$

The dominant energy condition (11) is equivalent to demanding that the energy density be nonnegative and the energy flow causal. All known forms of matter satisfy this condition. For a perfect fluid is reduces to $\rho \geq |p|$.

The strong energy condition for a perfect fluid reduces to the usual requirement that $p+3p \geq 0$, i.e., a large negative energy density or large negative pressures must be present to violate this condition.

By considering the field equations (7), we examine these properties (for the $\Lambda = 0$ case) in those models for which the metric is represented in diagonal form, namely

$$ds^2 = dt^2 + g_{11}(t)(\omega^1)^2 + g_{22}(t)(\omega^2)^2 + g_{33}(t)(\omega^3)^2 \tag{12}$$

Here ω^i $(i = 1, 2, 3)$ are one-forms. Also, we take

$$T_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} - pg_{\mu\nu} \tag{13}$$

with

$$p = h\rho, \qquad 0 \leq h \leq 1, \tag{14}$$

where h is a constant. The streamlines of the fluid are timelike geodesics with the tangent vector u_{μ}.

Now, if we introduce the scale factor

$$s(t) = (g_{11} \cdot g_{22} \cdot g_{33})^{1/6} \tag{15}$$

then

$$\Box R = \ddot{R} + 3(\dot{s}/s)\dot{R} \tag{16}$$

where a dot denotes a derivative with respect to the time t.

4. TIMELIKE GEODESICS

Suppose the matter is at rest in the coordinate system under consideration. Then

$$u^\mu = \delta_0^\mu \tag{17}$$

Therefore, by multiplying both sides of equation (7) by $\delta_0^\mu \delta_0^\nu$, we get

$$\begin{aligned}(1+2a_2R+3a_3R^2+4a_4R^3+\cdots+Na_NR^N)R_{00}\\ = \kappa\rho+R/2+\tfrac{1}{2}(a_2R^2+a_3R^3+a_4R^4+\cdots+a_NR^N)\\ -(3\dot{s}\dot{R}/s)[2a_2+2\cdot 3a_3R+3\cdot 4a_4R^2+\cdots+(N-I)Na_NR^{N-2}]\\ +(R_{;\alpha}R^{;\alpha}-R^2)[1\cdot 2\cdot 3a_3+2\cdot 3\cdot 4a_4R+3\cdot 4\cdot 5a_5R^2\\ +\cdots+(N-2)(N-1)Na_NR^{N-3}]\end{aligned} \tag{18}$$

by virtue of (13) and (16)

The timelike geodesics are complete if

$$R_{00}<0 \tag{19}$$

Therefore, in view of (8), the timelike geodesics are complete; then, for

$$(1+2a_2R+3a_3R^2+4a_4R^3+\cdots+Na_NR^{N-1})>0 \quad \text{or} \quad <0 \tag{20}$$

the following conditions must hold:

$$\begin{aligned}\{\kappa\rho+R/2+\tfrac{1}{2}(a_2R^2+a_3R^3+\cdots+a_NR^N)\\ -(3\dot{s}\dot{R}/s)(2a_2+2\cdot 3a_3R+3\cdot 4a_4R^2+\cdots+(N-1)Na_NR^{N-2})\\ -(R_{;\alpha}R^{;\alpha}-R^2)[1\cdot 2\cdot 3a_3+2\cdot 3\cdot 4a_4R+3\cdot 4\cdot 5a_5R^2+\cdots\\ +(N-2)(N-1)Na_NR^{N-3}]\}<0 \quad \text{or} \quad >0\end{aligned} \tag{21}$$

where obviously $\rho = T_{00}$.

Here it should be noted that for $N=2$ (the Lagrangian becomes $R+a_2R^2$), the conditions (19)-(21) for completeness of timelike geodesics are $R_{00}<0$ leading to

$$\text{for} \quad 1+2a_2R>0: \qquad \kappa\rho+R/2+\tfrac{1}{2}a_2R^2-6a_2\dot{s}\dot{R}/s<0 \tag{22}$$

$$\text{for} \quad 1+2a_2R<0: \qquad \kappa\rho+R/2+\tfrac{1}{2}a_2R^2-6a_2\dot{s}\dot{R}/s>0. \tag{23}$$

5. NULL GEODESICS

Here we consider the case of radial null geodesics only. The tangent vector is taken as

$$\omega^\mu = (l^0, l', 0, 0) \tag{24}$$

with

$$l_\mu l^\mu = 0 \tag{25}$$

Therefore, multiplying both the sides of equation (7) by $\omega^\mu \omega^\nu$, we get

$$\begin{aligned}
&(1+2a_2R+3a_3R^2+4a_4R^3+\cdots+Na_NR^{N-1})R_{\mu\nu}\omega^\mu\omega^\nu \\
&\quad =\kappa T_{\mu\nu}\omega^\mu\omega^\nu+[1\cdot 2a_2+2\cdot 3a_3R+3\cdot 4a_4R^2 \\
&\qquad +\cdots+(N-1)Na_NR^{N-2}]R_{;\mu;\nu}\omega^\mu\omega^\nu \\
&\qquad +[1\cdot 2\cdot 3a_3+2\cdot 3\cdot 4a_4R+3\cdot 4\cdot 5a_5R^2 \\
&\qquad +\cdots+(N-2)(N-1)Na_NR^{N-3}][R_{;\mu}R_{;\nu}\omega^\mu\omega^\nu]
\end{aligned} \tag{26}$$

These geodesics are complete provided that

$$R_{\mu\nu}\omega^\mu\omega^\nu < 0 \tag{27}$$

This means that for

$$1+2a_2R+3a_3R^2+4a_4R^3+\cdots+Na_NR^{N-1} \gtrless 0 \tag{28}$$

the following conditions must hold accordingly:

$$\begin{aligned}
&\chi T_{\mu\nu}\omega^\mu\omega^\nu+[1\cdot 2a_2+2\cdot 3a_3R+\cdots+(N-1)Na_NR^{N-2}]R_{;\mu;\nu}\omega^\mu\omega^\nu \\
&\quad +[1\cdot 2\cdot 3a_3+2\cdot 3\cdot 4a_4R+\cdots+(N-2)(N-1)Na_NR^{N-3}]R_{;\mu}R_{;\nu}\omega^\mu\omega^\nu \\
&\quad \gtrless 0
\end{aligned} \tag{29}$$

These conditions could be further simplified by using equations (24) and (25).

However, for $N=2$, these geodesics are complete if they satisfy (27). This leads to

$$\text{for}\quad 1+2a_2R>0: \qquad \tfrac{2}{3}\chi\rho+R/6-a_2[(3\dot{s}/s)+(\dot{g}_{11}/2g_{11})]\dot{R}<0, \tag{30}$$

$$\text{for}\quad 1+2a_2R<0: \qquad \tfrac{2}{3}\chi\rho+R/6-a_2[(3\dot{s}/s)+(\dot{g}_{11}/2g_{11})]\dot{R}>0, \tag{31}$$

where $\rho=T_{00}$. For the empty case $\rho=0$, one finds

$$R-6a_2\dot{R}[(3\dot{s}/s)+(\dot{g}_{11}/2g_{11})]<0 \quad \text{or} >0 \tag{32}$$

depending on whether $1+2a_2R>0$ or <0.

For completeness of the timelike and null geodesics there are formal restrictions, as seen in the above. Are they satisfied? They offer the opportunity of avoiding singularities.

6. SOME CONSIDERATIONS FOR $N=2$

Here the action can be written as

$$A=-(1/16\pi)\int d^4x\,(-g)^{1/2}(R+a_2R^2)+\text{surface terms} \tag{33}$$

and the conditions for the completeness of timelike and null geodesics given in the above. Furthermore, it can be seen that in the FRW universe (Le Denmat and Sirousse Zia, 1987), timelike convergence leads to $\ddot{s}\leq 0$. Thus, one needs to find a solution with $\ddot{s}>0$, which is seen to be the definition of generalized inflation (Lucchin and Matarrease, 1985). But, for a null geodesic, $R_{\mu\nu}\omega^\mu\omega^\nu<0$ leads to $(\dot{s}/s)$. This means that the Hubble parameter increases with time.

Whitt (1984) has expressed the full fourth-order field equations obtainable from the action (33) as Einstein gravity coupled to a massive scalar field. So we define a new metric by

$$\tilde{g}_{\mu\nu}=(1+2a_2R)g_{\mu\nu} \tag{34}$$

Then

$$\begin{aligned}\tilde{R}_{\mu\nu}=R_{\mu\nu}&-[(2a_2R_{;\mu;\nu}+g_{\mu\nu}a_2\square R)/(1+2a_2R)]\\&+[(6a_2^2R_{;\mu}R_{;\nu})/(1+2a_2R)^2]\end{aligned} \tag{35}$$

where

$$\phi=(3/4\pi)^{1/2}a_2R \tag{36}$$

This leads to the new field equations

$$\tilde{R}_{\mu\nu}-\tfrac{1}{2}\tilde{g}_{\mu\nu}\tilde{R}=8\pi\tilde{T}_{\mu\nu} \tag{37}$$

where

$$\tilde{T}_{\mu\nu}=[1+4(\pi/3)^{1/2}\phi]^{-2}\{\phi_{;\mu}\phi_{;\nu}-\tfrac{1}{2}\tilde{g}_{\mu\nu}[\phi_{;\lambda}\phi^{;\lambda}+(1/6a_2)\phi^2]\} \tag{38}$$

that is, we have the energy-momentum tensor now multiplied by a factor $[1+4(\pi/3)^{1/2}\phi]^{-2}$. If ϕ is small, gravity is coupled to a scalar field with $m^2=(6a_2)^{-1}$. But when $\phi=-1/4(3/\pi)^{1/2}$, the conformal transformation relating $\tilde{g}_{\mu\nu}$ and $g_{\mu\nu}$ is singular.

The conditions for geodesic completeness in the null timelike cases as discussed in the above can also be studied here for the expression (38) of $\tilde{T}_{\mu\nu}$.

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